Random processes and Monte Carlo Simulation

- Applications of Monte Carlo simulations.
- Random number generators.

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- Coin toss: the only prediction about the outcome: 50% of the events will end up as tail being up.
- Dice: In a large number of throws, the probability of getting a given face is $\frac{1}{6}$.

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- Often need a program that generates a random variable with a given probability distribution.

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- Hardware – external sources of random numbers which are generated by from a physical process.
- Software – source of pseudo random numbers.

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- This is because computers have only a limited number of bits to represent a number.
- It implies that no matter which pseudo random number generator you use – it will always repeat itself (period of the generator).

Good random number generators

Important issues:

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- Feedback shift register methods

Linear congruent method for RNG

Generates a pseudo random sequence of numbers $\{x_1, x_2, \ldots, x_k\}$ of length M over the interval $[0, M-1]$:

$$x_i = \mod (ax_{i-1} + c, M) = \text{remainder}\left(\frac{ax_{i-1} + c}{M}\right)$$

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Note that

$$\mod (b, M) = b - \text{int}(b/M) * M$$

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interval:$0 - 8$ i.e. $[0, M - 1]$
Period: 9 i.e. M numbers (then repeat).

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- For $c = 0$ called "Multiplicative congruential generator".

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- Scale results from y_i on $[0, 1]$ to z_i on $[A, B]$

$$z_i = A + (B - A) * y_i$$

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Simple shift register where the vacated bit is filled with the exclusive-or of two other bits in the shift register.

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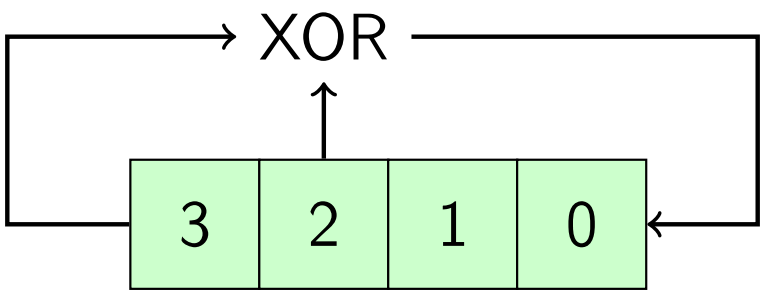

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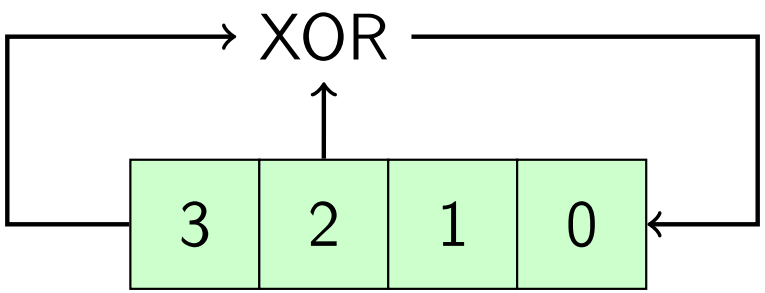

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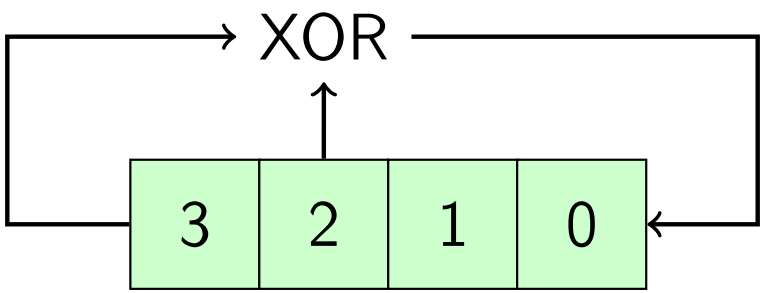

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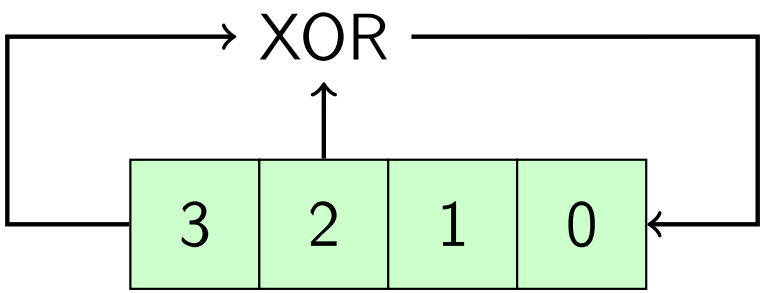

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- The register is shifted 1 step to the left.
- The result of the exclusive-or is entered into bit 0.

4bit shift register PRNG

Here is the pattern of bits, starting with 0001:

0001
0010
0100
1001
0011
0110
1101
1010
0101
1011
0111
1111
1110
1100
1000
0001

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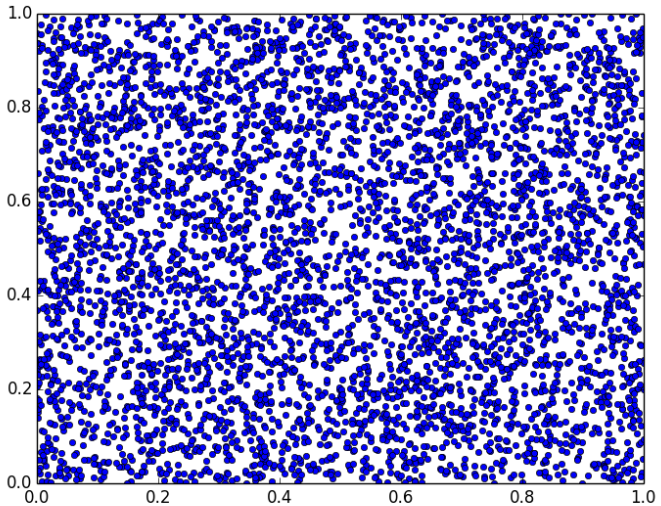
- Implemented in numpy.

How do we check the RNG?

- 2D plot, x_i and y_i from two random sequences (parking lot test).



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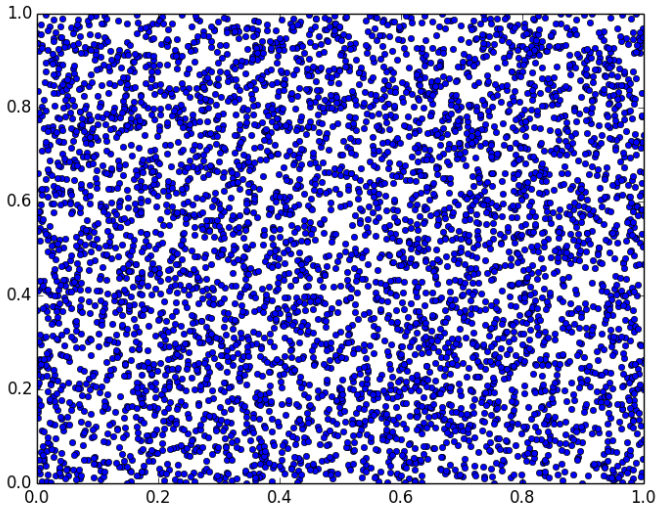

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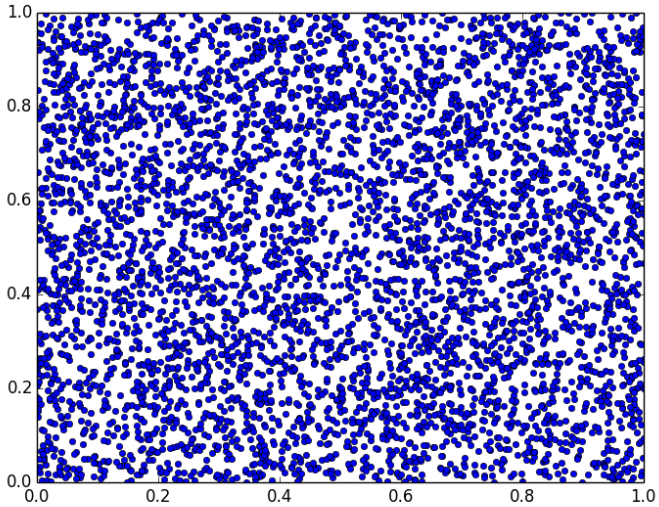
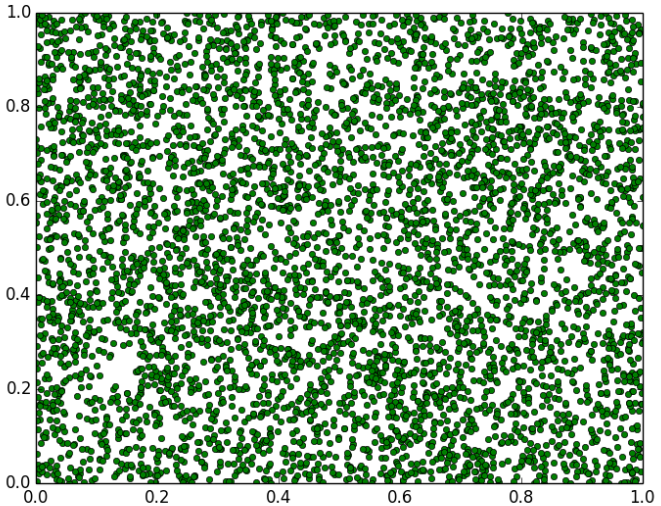


- Plot 3D figure (x_i, y_i, z_i)
- Plot correlation (x_i, x_{i+k})



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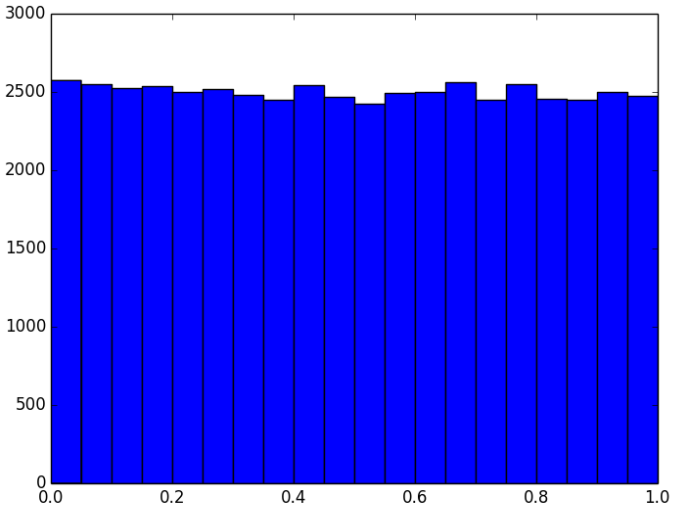
- Near neighbour correlation:

$$\frac{1}{N} \sum_{i=1}^{N} x_i x_{i+k} \approx \frac{1}{4}$$

Examples of other assessments

- Uniformity (50000 random numbers)



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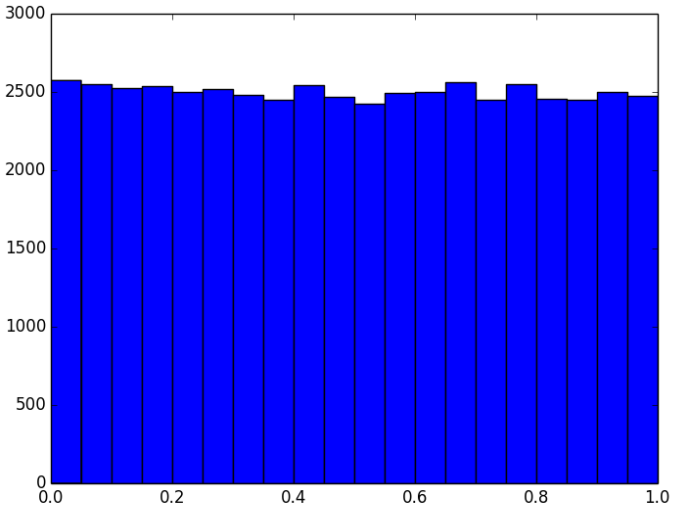

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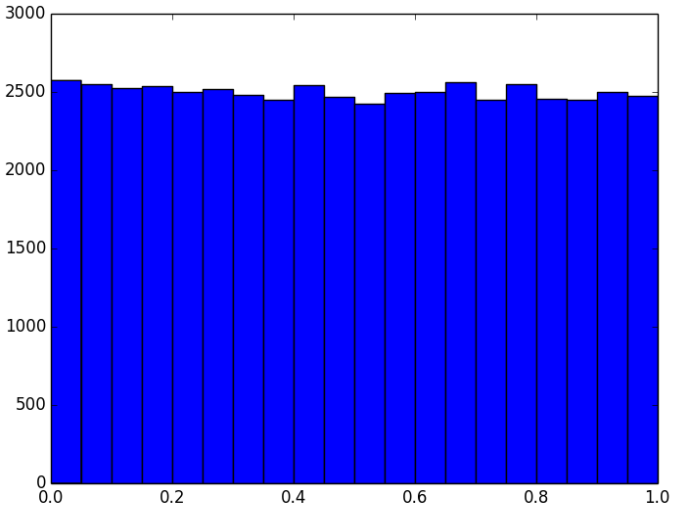

- 4th moment: (50000 random numbers)

$$< x^4 >= 0.1988$$

- near neighbor correlation: (50000 random numbers) $= 0.2478$

Test suites for RNG

Good test suites exist – TestU01 – which can be used to uncover problems in random number generators.

Dont try to invent your own random number generator – unless you know what you are doing. This is very tricky business!!!