- Discrete Cosine Transform.
- Fast Fourier Transform.
- Convolution.
- Power spectrum.

Discrete Cosine Transform

- If the function $f(x)$ is even (i.e. symmetric) about the midpoint $(x = \frac{L}{2})$ then one can write the cosine series:

$$f(x) = \sum_{k=0}^{\infty} \alpha_k \cos(\frac{2\pi kx}{L})$$

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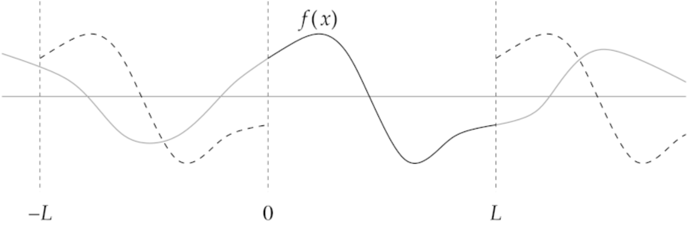
- This might seem like a big limitation making the whole cosine transform virtually useless – but this is not the case.

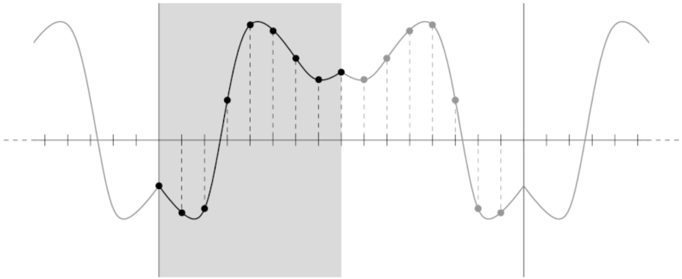
Discrete Fourier Transform – reminder

- We had made any function periodic – say if we are only interested in a portion of this non periodic function over a finite interval, 0 to L, we can just take that portion and repeat it to create a periodic function.



Discrete Cosine Transform – even functions

- If we are interested in the function in a finite region, we can make it symmetric by adding to it a mirror image of itself and then repeating it endlessly!



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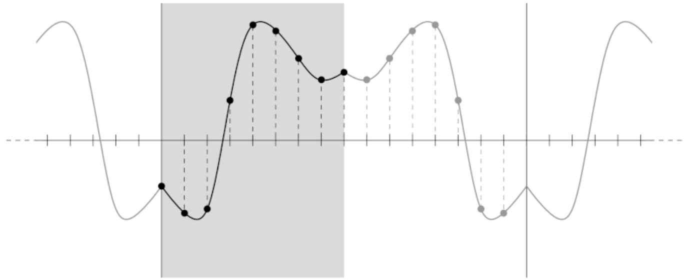
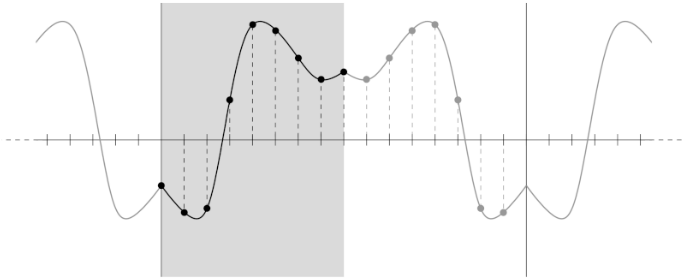


Discrete Cosine Transform – even functions

- If we are interested in the function in a finite region, we can make it symmetric by adding to it a mirror image of itself and then repeating it endlessly!
- In practice, this is how the cosine transform is always used.
- This also implies that the number of samples in the transform is always even.



Discrete Cosine Transform (DCT)

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&= \sum_{n=0}^{\frac{1}{2}N} y_n \exp(-i\frac{2\pi kn}{N}) + \sum_{n=\frac{1}{2}N+1}^{N-1} y_{N-n} \exp(i\frac{2\pi k(N-n)}{N})
\end{aligned}
$$

Because the function is symmetric $y_0 = y_N, y_1 = y_{N-1}, \ldots$ and $e^{i2\pi k} = 1$ for all $k \in \mathbb{Z}$

Changing variables $N - n \to n$ in the right hand expression:

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Normally the cosine transform is applied to real samples, which implies that the coefficients c_k will all be real as well (as they are sums of real terms).

As y_n and c_k are real, $c_{N-r} = c_r^* = c_r$. The inverse transform:

$$y_n = \frac{1}{N}\left[\sum_{k=0}^{N-1} c_k \exp(i\frac{2\pi kn}{N})\right]$$

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- A nice feature of this DCT is that unlike DFT, it does not assume that the samples are periodic.
- This is much better suited for non periodic functions as there is no discontinuity introduced.
- In principle, the discrete sine transform can also be computed. However, the requirement of anti-symmetry forces the function to be zero at either end of the range. This does not happen often in real-world applications...

Fast Fourier Transform

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- Since there are N c_k's so the total number of operations is $\mathcal{O}(N^2)$.
- Gauss came up with a trick to reduce the number of operations. Often the FFT is attributed to Cooley and Tukey – but Gauss used it in 1805 (when he was 28 yrs).

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&\equiv E_k + e^{-i2\pi k/N} O_k
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However we also know:

$$\begin{aligned} e^{-i2\pi(k+\frac{1}{2}N)/N} &= e^{-i2\pi k/N - i\pi} \\ &= e^{-i\pi} e^{-i2\pi k/N} \\ &= -e^{-i2\pi k/N} \end{aligned}$$

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- The algorithm gains its speed by re-using the results of intermediate computations to compute multiple DFT outputs.
- From the original N^2 multiplications, now one has $2(\frac{N}{2})^2 = \frac{N^2}{2}$ multiplications.
- This procedure can be recursively repeated – leading to a scaling of $\mathcal{O}(N \log_2 N)$.

Fast Fourier Transform

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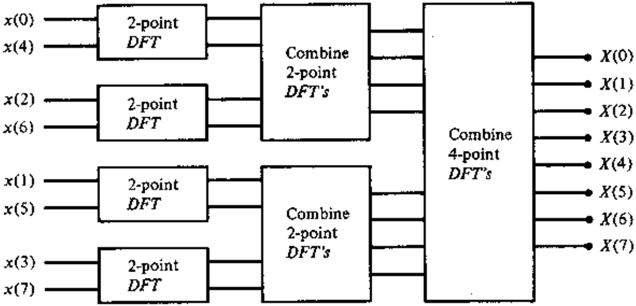
Fast Fourier Transform



Convolution

$$
\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau
\end{aligned}
$$

The convolution theorem states that the Fourier transform of a convolution of two functions is the pointwise product of their Fourier transforms.

$$
\begin{aligned}
\mathcal{F}\{f * g\} &= \mathcal{F}\{f\} \cdot \mathcal{F}\{g\} \\
\implies f * g &= \mathcal{F}^{-1}\big\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\big\}
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- It can be shown that the inverse transform of the periodogram is the sample autocorrelation function.
- Parseval's theorem tells us:

$$\sum_{n=0}^{N-1} |y_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |c_k|^2$$